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# THE PROPERTIES OF SOLUTIONS OF THE FUNDAMENTAL PROBLEM OF DYNAMICS IN SYSTEMS WITH NON-IDEAL CONSTRAINTS†

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The problem of determining the accelerations and reactions of constraints in systems with Coulomb friction is discussed. It is shown that a realistic solution of the problem is possible provided only that allowance is made for deformations in the bodies comprising the system. For this purpose, the initial system is expanded by including local deformations among the generalized coordinates. Asymptotic methods are used to divide the expanded system into a "slow" initial subsystem and a "fast" subsystem that serves to determine the reactions. Analysis of the fast subsystem is the key to understanding the dynamics on the whole. The results obtained for the number of steady solutions and their stability are invariant with respect to the viscoelastic characteristics of the contact pairs. For every stable steady solution there is a realizable motion of the initial system, but in order to select the true motion one has to know the initial deformations and their derivatives with respect to time. Along with steady solutions, the "fast" subsystem may have stable oscillatory solutions. It is proved that to the stable limit cycle of the "fast" subsystem there correspond motions of the initial system in which the reactions oscillate at a high frequency about their mean values. The Painlevé–Klein system and the problem of braking a wheel with two brake shoes are considered as examples. 

Previous work in which allowance is made for deformations and fast motions has been widely used [1–5] to investigate systems with one friction pair in connection with solution of the Painlevé paradoxes [6]; cases in which the fast subsystem has a unique stable equilibrium or no such equilibria (a catastrophe of the impact type) have been discussed. Examples of oscillations in the acoustic frequency range in systems with several friction pairs that have been considered are the "howling" of pistons [7] and brake shoe "squeal" [8].

## 1. THE EQUATIONS OF MOTION TAKING DEFORMATIONS INTO ACCOUNT

Suppose the configuration of a system of rigid bodies is described by coordinates $\mathbf{q} \in R^n$. The displacements are subject to certain constraints, which are not generally ideal. We shall assume that the constraints are described by the reactions

$$q_j = 0, \quad j = 1, \dots, k \tag{1.1}$$

Each of relations (1.1) corresponds to the contact of a pair of bodies. From a geometrical standpoint, it defines a plane containing the vectors of the possible relative displacements of the bodies, as well as the tangential components $\mathbf{T}^{(j)}$ of the reaction $\mathbf{R}^{(j)}$. The direction of the normal component of the reaction is that of a unit vector $\mathbf{n}_j$ orthogonal to the plane in the sense of the Jacobi metric defined by the kinetic energy. Thus,

$$\mathbf{R}^{(j)} = \mathbf{T}^{(j)} + N_j\mathbf{n}_j, \quad j = 1, \dots, k$$

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When the equations of motion are formulated, the constraints (1.1) are not taken into consideration, but their reactions are added to the active forces. Using the fundamental theorems of dynamics, we can write these equations in the form

$$\ddot{\mathbf{q}} = \mathbf{F}(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{R}^{(1)}, \ldots, \mathbf{R}^{(k)}), \quad \mathbf{q}, \mathbf{F}, \mathbf{R}^{(j)} \in R^n \tag{1.2}$$

To describe the friction, we use laws of the form

$$\mathbf{T}^{(j)} = \mathbf{T}_j(\mathbf{q}, \dot{\mathbf{q}})N_j, \quad j = 1, \ldots, k \tag{1.3}$$

It is assumed that at the instant of time under consideration the velocities of relative slippage at all contacts are not zero.

Substituting expressions (1.3) into Eqs (1.2), we obtain a system

$$\ddot{\mathbf{q}} = \mathbf{G}(t, \mathbf{q}, \dot{\mathbf{q}}, N_1, \ldots, N_k) \tag{1.4}$$

where the function $\mathbf{G}$ is piecewise continuously differentiable in some domain that includes the initial values of the variables.

In classical mechanics, system (1.4) is combined with equalities (1.1) to solve the fundamental problem of dynamics – to determine the generalized accelerations $\ddot{\mathbf{q}}$ and the normal reaction $N_j$. One drawback of that approach is that it is not possible to describe high-frequency oscillations in directions normal to the surfaces (1.1); but such oscillations may sometimes be of considerable practical value [7–9]. In addition, in systems with dry friction one may obtain paradoxical situations in which system (1.1), (1.2) in unsolvable or has several solutions [6].

To eliminate these drawbacks, some (or all) of the constraints (1.1) will be dropped. From a physical standpoint, this is equivalent to changing from absolutely rigid bodies to deformable ones. Since the addition of degrees of freedom complicates the analysis, allowance is made in each specific case for only the most significant deviations from the absolutely rigid model, taking the physical properties of the bodies that comprise the system into consideration.

*Example.* Two masses connected by a weightless rod can move along parallel guides. This system has been proposed [6] to demonstrate dry friction paradoxes. Here there are three constraints, expressing the fact that the masses cannot leave the guides, that the length of the rod is constant, and that a single coordinate $q$ describes the displacement of the masses along the guides. To eliminate the paradoxes, it has been assumed [1] that the rod is stretched, so that an additional degree of freedom appears. As a single additional variable it has also been suggested [5] that one should take the longitudinal displacement of one of the guides. One can obtain a system with two additional coordinates by assuming that the rod is rigid but the guide is pliable in the normal direction. Combining these assumptions, one can construct a model with three or more additional coordinates.

We shall assume henceforth that relations (1.1) may be violate by deformations and that the normal reactions are certain differentiable functions of the appropriate deformations and their velocities

$$N_j = N_j(q_j, \dot{q}_j), \quad \partial N_j/\partial q_j < 0, \quad \partial N_j/\partial \dot{q}_j < 0, \quad \partial N_j(0,0)/\partial q_j \neq 0; \quad j = 1, \ldots, k \tag{1.5}$$

Inequalities (1.5) express the increase in stresses together with deformations, as well as the presence of dissipation. The last group of inequalities means that the stiffness of the constraints for infinitesimal deformations is non-zero.

Substituting relations (1.5) into (1.4), we obtain a system of ordinary second-order differential equations. This model is not directly suitable for analysis, since the functions (1.5) are singular in nature in the sense of the estimates $|q_j| \ll 1$, $|\partial N_j/\partial q_j| \gg 1$. Hence small initial perturbations may turn out to have a significant effect on the nature of the solution of the system, and one cannot let $q_j \to 0$ in Eqs (1.4).

With the aim of regularization, we replace $q_j$ and $\dot{q}_j$ by new phase variables $u_j$ and $\upsilon_j$ ($j = 1, \ldots, k$), whose values will not vanish when taking the aforementioned limit.

In particular, we can put

$$u_j = N_j(q_j, \dot{q}_j), \quad \upsilon_j = \dot{q}_j/\varepsilon, \quad j = 1, \ldots, k \tag{1.6}$$

where $\varepsilon$ is a small parameter characterizing the rate of deformation for the given loads.

Because of the first inequality in (1.5), transformation (1.6) is invertible. Transforming Eqs (1.4) accordingly, we obtain

$$\ddot{q}_i = G_i, \quad \varepsilon\dot{u}_j = U_j, \quad \varepsilon\dot{v}_j = G_j$$
$$U_j = \varepsilon^2\frac{\partial N_j}{\partial q_j}v_j + \varepsilon\frac{\partial N_j}{\partial \dot{q}_j}G_j, \ \mathbf{G} = \mathbf{G}(t, q_j(u_j, \varepsilon v_j), q_i, \varepsilon v, \dot{q}_i); \ j = 1, \dots, k; \ i = k+1, \dots, n \tag{1.7}$$

By the third group of inequalities in (1.5), we have

$$\dot{q}_j = O(\varepsilon), \quad q_j = O(\varepsilon^2); \quad j = 1, \dots, k$$

whence it follows that

$$\partial N_j/\partial q_j = O(\varepsilon^{-2}), \quad \partial N_j/\partial \dot{q}_j = O(\varepsilon^{-1}) \ j = 1, \dots, k \tag{1.8}$$

By (1.8), the right-hand sides of system (1.7) are bounded as $\varepsilon \to \infty$. Transforming to the "stretched" time $\tau = t/\varepsilon$, we obtain

$$\mathbf{x}' = \varepsilon\mathbf{X}(\mathbf{x}, \mathbf{y}, \varepsilon), \quad \mathbf{y}' = \mathbf{Y}(\mathbf{x}, \mathbf{y}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{y}(0) = \mathbf{y}_0$$
$$\mathbf{x} = (\mathbf{t}, \mathbf{q}_i, \dot{\mathbf{q}}_i), \quad \mathbf{y} = (\mathbf{u}_j, \mathbf{v}_j), \quad \mathbf{X} = (1, \mathbf{x}_{n+i}, \mathbf{G}_i), \quad \mathbf{Y} = (\mathbf{U}_j, \mathbf{G}_j) \tag{1.9}$$
$$j = 1, \dots, k; \quad i = k+1, \dots, n$$

where the prime denotes differentiation with respect to $\tau$.

The symbols $G_i$, $U_j$ and $G_j$ in system (1.9) denote piecewise continuously differentiable functions of the phase variables of the parameter $\varepsilon$ and $t$. The first group of equations describes "slow" variation of the phase variables in the classical model, while the remaining equations describe "fast" motions; in the classical approach, the latter are ignored, and the differential equations are replaced by the algebraic equations (1.1).

*Example.* A point mass of unit mass is sliding along a rough guide under the action of a two-dimensional system of forces. We introduce a system of coordinates $XOY$ in such a way that the guide lies along the abscissa axis and the initial velocity $\dot{x}$ of the point is positive. Equations (1.2) become

$$\ddot{x} = -\mu|N| + X(x, \dot{x}), \quad \ddot{y} = N + Y(x, \dot{x}) \tag{1.10}$$

which $\mu$ is the coefficient of friction. We assume that the function (1.5) is linear:

$$N = -2bc\dot{y} - c^2 y, \quad c \gg 1 \tag{1.11}$$

Put $\varepsilon = 1/c$ and change in Eqs (1.10) and (1.11) to the variables $u = N$, $v = c\dot{y}$, $q_2 = x$, $p_2 = \dot{x}$ and the "stretched" time $\tau = ct$. We have

$$q_2' = \varepsilon p_2, \ p_2' = \varepsilon(X(q_2, p_2) - \mu|u|), \ u' = -v - 2bu - 2bY(q_2, p_2), \ v' = u + Y(q_2, p_2) \tag{1.12}$$

If $\varepsilon = 0$, the variables $q_2$ and $p_2$ will not vary with time and all the trajectories in the phase plane of fast motions $(u, v)$ will tend to the equilibrium position $u = -Y$, $v = 0$ as $\tau \to \infty$, since the eigenvalues of the matrix of this linear subsystem are negative. Substituting the value $u = -|Y|$ into the first two equations of (1.12) and returning to the original time $t = \varepsilon\tau$, we obtain the well-known classical equation

$$\ddot{x} = X - \mu|Y|$$

In this example, all the fast motions approach a global attractor. Therefore, expansion of the classical model by admitting deformations is not desirable. Nevertheless, one cannot exclude *a priori* more complicated cases of fast dynamics, in which there are several attractors or that they have a complicated structure. In such cases it is not admissible to ignore the fast motions.

A systematic approach to the analysis of system (1.9) will be proposed below.

## 2. ASYMPTOTIC SEPARATION OF MOTIONS

We shall study system (1.9) using methods of asymptotic separation of motions [10–12]. Let us consider the associated system

$$\mathbf{y}' = \mathbf{Y}(\mathbf{x}, \mathbf{y}, 0), \quad \mathbf{y}(0) = \mathbf{y}_0 \tag{2.1}$$

The values of $\mathbf{x}$ in Eqs (2.1) are "frozen", playing the role of parameters. System (2.1) is of lower dimensions than (1.9), so it is easier to analyse. The aim of the analysis is to look of attractors: attractive equilibrium positions, limit cycles, and the like. Bearing in mind that the variables $\mathbf{x}$ in system (1.9) vary with time, we require the attractors to be structurally stable. In the case of equilibrium positions and limit cycles, it is sufficient to that end to require stability in the first approximation.

Let us assume that at $\mathbf{x} = \mathbf{x}_0$ system (2.1) has an equilibrium position $\mathbf{y} = \mathbf{y}^*$, and that all the eigenvalues of the Jacobi matrix

$$\mathbf{J}_0 = \mathbf{J}(\mathbf{x}_0) = \|\partial\mathbf{Y}(\mathbf{x}_0, \mathbf{y}^*, 0)/\partial\mathbf{y}\|$$

have negative real parts. By the implicit function theorem, there is a neighbourhood of the point $(\mathbf{x}_0, \mathbf{y}^*)$ in which one can construct an equilibrium surface $\mathbf{y} = \bar{\mathbf{y}}(\mathbf{x})$ of class $C_1$, at whose points $\mathbf{Y}(\mathbf{x}, \bar{\mathbf{y}}(\mathbf{x}), 0) = 0$, and moreover $\mathbf{y}^* = \bar{\mathbf{y}}(\mathbf{x}_0)$.

Define a comparison system by

$$\dot{\bar{\mathbf{x}}} = \varepsilon\mathbf{X}(\bar{\mathbf{x}}, \bar{\mathbf{y}}(\bar{\mathbf{x}}), 0), \quad \bar{\mathbf{x}}(0) = \mathbf{x}_0 \tag{2.2}$$

The phase variables in system (2.2) are the generalized coordinates and velocities of the original system (1.1), but deformations of the constraints are excluded: when one returns to the original independent variable, the parameter $\varepsilon$ disappears.

The following well-known proposition [11, 12] will be proved below by a method that will enable us to obtain certain further generalizations.

*Proposition* 1. Positive numbers $\varepsilon_1$, $T$, $C$ and $\delta$ exist such that, for all $\varepsilon$, $\tau$ and $\mathbf{y}_0$ that satisfy the conditions

$$0 < \varepsilon \leq \varepsilon_1, \quad 0 < \tau \leq T/\varepsilon, \quad \|\mathbf{y}_0 - \mathbf{y}^*\| < \delta$$

the solutions of systems (1.9) and (2.2) are close together in the sense of the inequality

$$\|\mathbf{x}(\tau) - \bar{\mathbf{x}}(\tau)\| + \|\mathbf{y}(\tau) - \bar{\mathbf{y}}(\bar{\mathbf{x}})\| \leq C\varepsilon \tag{2.3}$$

*Proof.* Under the above assumptions, one can construct a Lyapunov function for system (2.1) as a quadratic form in $\mathbf{w} = \mathbf{y} - \bar{\mathbf{y}}(\mathbf{x})$

$$V(\mathbf{w}) = 1/2(\mathbf{B}\mathbf{w}, \mathbf{w}), \quad V' = (\mathbf{B}\mathbf{w}, \mathbf{J}(\mathbf{x})\mathbf{w}) + O(\|\mathbf{w}\|^3) \leq -C_1 w^2 \tag{2.4}$$

where $C_1 = \text{const} > 0$, $\mathbf{B} = \mathbf{B}(\mathbf{x})$ is a symmetric positive matrix. Inequality (2.4) will be satisfied for fairly small values of $\|\mathbf{w}\|$ and $\|\mathbf{x} - \mathbf{x}_0\|$.

Note that if $\varepsilon \neq 0$, then the value $\mathbf{w} = \mathbf{0}$ will not satisfy system (1.9). Nevertheless, the function (2.4) may serve as a measure of the deviation of the "fast" variables from their equilibrium values. The derivative of this function along trajectories of Eqs (1.9) is

$$V' = \left(\mathbf{B}\mathbf{w}, \mathbf{Y}(\mathbf{x}, \mathbf{w} + \bar{\mathbf{y}}(\mathbf{x}), \varepsilon) - \varepsilon\frac{\partial\bar{\mathbf{y}}}{\partial\mathbf{x}}\mathbf{X}(\mathbf{x}, \mathbf{w} + \bar{\mathbf{y}}(\mathbf{x}), \varepsilon)\right) + O(\|\mathbf{w}\|^3) \leq -C_2 w^2 + C_3\varepsilon\|\mathbf{w}\| \tag{2.5}$$

Inequality (2.5) will hold for certain constants $C_2$ and $C_3$ if the numbers $\varepsilon$ and $\|\mathbf{x} - \mathbf{x}_0\|$ are sufficiently small and all the eigenvalues of the characteristic matrix

$$\mathbf{J}(\mathbf{x}) = \|\partial\mathbf{Y}(\mathbf{x}, \bar{\mathbf{y}}(\mathbf{x}), 0)/\partial\mathbf{y}\|$$

have negative real parts. The number $\|\mathbf{x} - \mathbf{x}_0\|$ may be made small by suitable choice of the constant $T$ (which does not depend on $\varepsilon$).

By inequality (2.5), $V'$ is negative in the spherical layer

$$C_4 \geq \|\mathbf{w}\| \geq \varepsilon C_3/C_2$$

where $C_4$ is a constant. Thus, if this layer contains the level surface $V(\mathbf{w}) = V(\mathbf{w}_0)$, the inequality $V(\mathbf{w}) \leq V(\mathbf{w}_0)$ will hold in the time interval where inequality (2.5) holds.

To estimate the magnitude of the deviation $\boldsymbol{\alpha}(\tau) = \mathbf{x}(\tau) - \bar{\mathbf{x}}(\tau)$, it will suffice to change to integral equations

$$\mathbf{x}(\tau) = \mathbf{x}_0 + \varepsilon\int_0^\tau \mathbf{X}(\mathbf{x}(s), \mathbf{y}(s), \varepsilon)ds, \quad \bar{\mathbf{x}}(\tau) = \mathbf{x}_0 + \varepsilon\int_0^\tau \mathbf{X}(\bar{\mathbf{x}}(s), \bar{\mathbf{y}}(s), 0)ds$$

Hence it follows that

$$\alpha(\tau) \leq C_4\varepsilon^2\tau + C_5\varepsilon\int_0^\tau \alpha(s)ds \tag{2.6}$$

By Gronwall's Lemma

$$\alpha(\tau) \leq C_4\varepsilon^2\tau\exp(C_5\varepsilon\tau)$$

which implies the estimate (2.3).

Let us assume now that at $\mathbf{x} = \mathbf{x}_0$ system (2.1) has a periodic solution $\mathbf{y} = \mathbf{\Gamma}_0(\tau)$ which is stable in the first approximation. Then for values of $\mathbf{x}$ close enough to $\mathbf{x}_0$, it will also have a periodic solution $\mathbf{y} = \mathbf{\Gamma}(\mathbf{x}, \tau)$ close to $\mathbf{\Gamma}_0$. Denote the period of this solution by $T(\mathbf{x})$.

We define a comparison system by averaging the right-hand sides along the trajectory $\mathbf{\Gamma}(\mathbf{x}, \tau)$. We have

$$\bar{\mathbf{x}}' = \varepsilon\langle\mathbf{X}(\bar{\mathbf{x}})\rangle, \quad \bar{\mathbf{x}}(t_0) = \mathbf{x}_0, \quad \langle\mathbf{X}(\bar{\mathbf{x}})\rangle = \frac{1}{T(\bar{\mathbf{x}})}\int_0^{T(\bar{\mathbf{x}})} \mathbf{X}(\bar{\mathbf{x}}, \mathbf{\Gamma}(\bar{\mathbf{x}}, \tau), 0)d\tau \tag{2.7}$$

As in the case of comparison system (2.2), Eqs (2.7) do not include deformations.

*Proposition* 2. Positive numbers $\varepsilon_1$, $T$, $C$ and $\delta$ exist such that, for all $\varepsilon$, $\tau$ and $\mathbf{y}_0$ satisfying the conditions

$$0 < \varepsilon \leq \varepsilon_1, \quad 0 < \tau \leq T/\varepsilon, \quad \|\mathbf{y}_0 - \mathbf{\Gamma}(\mathbf{x}_0, 0)\| < \delta$$

the solutions of systems (1.9) and (2.7) are close together in the sense that

$$\|\mathbf{x}(\tau) - \bar{\mathbf{x}}(\tau)\| + \|\mathbf{y}(\tau) - \mathbf{\Gamma}(\bar{\mathbf{x}}(\tau), \tau)\| \leq C\varepsilon \tag{2.8}$$

The formal proof of this proposition is analogous to that of Proposition 1 for the discrete time case. The transition to a discrete system involves the construction of a Poincaré map in the neighbourhood of a period trajectory $\mathbf{\Gamma}(\mathbf{x}, \tau)$. One first constructs a quadratic Lyapunov function for $\varepsilon = 0$ and then uses it to estimate the smallness of the second term in (2.8). It then remains to apply Bogolyubov's theorem on averaging systems in standard form.

## 3. SYSTEM WITH ONE CONSTRAINT

We will now proceed to a qualitative analysis of system (2.1). In the simplest case, when $k = 1$ in (1.1), the phase space is two-dimensional. Typical attracting sets in such systems are equilibrium positions and limit cycles, and the results presented in Section 2 can be used.

We shall assume that the phase variables are defined by formulae (1.6), where $k = 1$. Equations (2.1) become

$$\begin{gathered} u' = \Phi(u, v)v + \Psi(u, v)f(u), \quad v' = f(u) \\ \Phi = \varepsilon^2\frac{\partial N_1}{\partial q_1}, \quad \Psi = \varepsilon\frac{\partial N_1}{\partial \dot{q}_1}, \quad f(u) = G_1(t_0, q_1(u), q_i, 0, \dot{q}_i) \end{gathered} \tag{3.1}$$

The equilibrium positions of system (3.1) are points of the form $(u^*, 0)$, where $f(u^*) = 0$. For the Jacobian at these points we have

$$\mathbf{J}_0 = \begin{Vmatrix} \Psi f'(u^*) & \Phi \\ f'(u^*) & 0 \end{Vmatrix}$$

Since $\Psi < 0$, $\Phi < 0$ because of the restrictions (1.5), we obtain the following.

*Proposition* 3. If

$$f(u^*) = 0, \quad f'(u^*) > 0 \tag{3.2}$$

then $(u^*, 0)$ is an equilibrium position of system (3.1) which is asymptotically stable in the first approximation. If the reverse inequality to (3.2) holds, the equilibrium is unstable.

By Proposition 1, for each root of the equation $f(u) = 0$ that satisfies condition (3.2) there is a solution of the complete system (1.9) that is stable in the sense of inequality (2.3). Note that this conclusion about the number and stability of equilibrium positions is invariant to the form of the function $N(q, \dot{q})$ (provided that conditions (1.5) hold). In particular, one can single out the following important special case.

*Proposition* 4. Consider system (1.2) with $k = 1$. If there are values of $t$, $\mathbf{q}$, $\dot{\mathbf{q}}$ such that the partial derivative $\partial F_1/\partial N_1$ exists for almost all $N_1$ and moreover

$$\partial F_1/\partial N_1 \geq c_1 > 0 \tag{3.3}$$

then the fundamental problem of dynamics for system (1.1), (1.2) has a unique steady solution, which is moreover stable.

Note that condition (3.3) is always satisfied in cases for which the constraint is ideal or in the case of viscous friction (the friction force is independent of the normal reaction). Indeed, reasoning from the Lagrange equations, one can represent Eqs (1.2) in the form

$$\ddot{\mathbf{q}} = \mathbf{A}^{-1}(N_1, 0, \ldots, 0)^T + \ldots$$

where $\mathbf{A}$ is the matrix of the quadratic part of the kinetic energy and the unwritten terms do not depend on $N_1$. Consequently, the quantity $\partial F_1/\partial N_1$ is equal to a corner element of the positive-definite matrix $\mathbf{A}^{-1}$.

Whether the problem admits of oscillatory solutions, corresponding to limit cycles of system (2.1), depends on the specific form of the function (1.5). We will confine ourselves to the case that is simplest and most frequently used in practical computations – when that function is linear (so that its partial derivatives are constant).

*Proposition* 5. If $\Phi$ and $\Psi$ in system (3.1) are constant, the system has no periodic solutions other than its equilibrium positions.

*Proof.* Let $\Pi(u)$ be some primitive function for $f(u)$. Consider the function

$$V(u, v) = 1/2\Phi v^2 - \Pi(u) \tag{3.4}$$

The derivative of this function along trajectories of system (3.1) has the form

$$V' = \Phi v v' - f(u)u' = -\Psi f^2(u)$$

If $f(u)$ is not identically equal to zero for some solution, then $V' < 0$, and this solution cannot be periodic, as required.

Let us sum up. In a system with one constraint, the dynamics is determined by the number of zeros of the function $f(u)$ in Eq. (3.1) for which inequality (3.2) holds. For each such value $u^*$ a realistic solution of the fundamental problem of dynamics exists, the absence of such values indicates a rapid, unbounded increase in the reaction $N$ (a catastrophe of the impact type). To choose the true solution from several possibilities, the initial deformations must be taken into account, as in Euler's classical example of the bifurcation of a loaded column.

*Example.* Let us formulate the equations of motion in the Painlevé–Klein example, assuming that the rod is deformable and that the guides are absolutely rigid. Suppose at a given instant of time the particles are sliding to the right. Assuming that they have unit mass, we can express the fundamental theorems of dynamics by the equations

$$\begin{aligned} \ddot{x}_1 &= X_1 - \mu_1|N_1| - R\cos\varphi, \quad \ddot{x}_2 = X_2 - \mu_2|N_2| + R\cos\varphi \\ N_1 &= R\sin\varphi - Y_1, \quad N_2 = -R\sin\varphi - Y_2 \end{aligned} \tag{3.5}$$

where $x_1$ and $x_2$ are the coordinates of the particles, $(X_1, Y_1)$ and $(X_2, Y_2)$ are the external forces applied to them, $R$ is the reaction of the rod (the inequality $R > 0$ means that the rod is stretched), $\mu_1$ and $\mu_2$ are the coefficients of friction, $\varphi$ is the angle between the rod and the guides (variations of this angle due to deformations will be ignored), and $2l$ is the length of rod. (An analogous system has been considered in the special case when $Y_1 = Y_2 = 0$ [1, 4, 6].)

Noting that $x_1 - x_2 = (2l - \delta)\cos\varphi$ and subtracting the first equality of (3.5) from the second, we obtain

$$\begin{aligned} \ddot{\delta} &= X^* + \mu_1^*|R - Y_1^*| - \mu_2^*|R + Y_2^*| + 2R \\ X^* &= (X_2 - X_1)/\cos\varphi, \quad \mu_{1,2}^* = \mu_{1,2}\mathrm{tg}\varphi, \quad Y_{1,2}^* = Y_{1,2}/\cos\varphi \end{aligned} \tag{3.6}$$

The right-hand side of Eq. (3.6) is identical with the function $f(u)$ in (3.1), where $u = R$. Let us find the zeros of this function as a function of the parameters of the problem. Put

$$\mu_\pm = \mu_1^* \pm \mu_2^* \equiv (\mu_1 \pm \mu_2)\mathrm{tg}\varphi$$

*The case* $Y_1^* + Y_2^* = 0$. We have

$$f(u) = X^* + \mu_-|u - Y_1^*| + 2u \tag{3.7}$$

The graph of the function (3.7) is a broken line, with segments of slopes $2 - \mu_+$ and $2 + \mu_-$ and break point $u_1 = Y_1^*, f_1 = X^* + 2Y_1^*$. The assumptions of Proposition 4 are expressed by the inequality

$$|\mu_-| < 2 \tag{3.8}$$

which is precisely the condition for the compatibility of the constraints obtained in [4] on the assumption that $Y_1^* = Y_2^* = 0$.

If $\mu_- > 2$, the break point of the line is a minimum. Thus, if $f_1 > 0$, the function (3.7) has no zeros (an impact catastrophe), but if $f_1 < 0$, it will have two zeros $u_{1,2}^*$, for which $u_1^* < u_1, f'(u_1^*) < 0$ (instability) and $u_2^* > u_1, f'(u_2^*) > 0$ (stability).

If $\mu_- < -2$, the break point of the line is a maximum. Thus, if $f_1 < 0$, the function (3.7) has no zeros (an impact catastrophe), but if $f_1 > 0$, it will have two zeros $u_{1,2}^*$, for which $u_1^* < u_1, f'(u_1^*) > 0$ (stability) and $u_2^* > u_1, f'(u_2^*) < 0$ (instability).

In terms of the theory of singularities, the violation of inequality (3.8) leads to a fold [13].

*The case* $Y_1^* + Y_2^* < 0$. The graph of the function (3.6) is a broken line with segments of slopes $2 - \mu_-$, $2 + \mu_+$ and $2 + \mu_-$. The results are analogous to those presented above: the validity of inequality (3.8) guarantees the existence of a unique – and stable – solution, but the reversed inequality attests either to an impact catastrophe or to the existence of a stable solution and an unstable one (a fold).

*The case* $Y_1^* + Y_2^* > 0$. The graph of the function $f(u)$ is a broken line whose segments have slopes $2 - \mu_-$, $2 + \mu_+$ and $2 + \mu_-$. If inequality (3.8) is reversed, two of these slopes (including the middle one) are negative and the third positive. By analogy with the previous case, we have a fold.

If

$$\mu_+ < 2 \tag{3.9}$$

the assumptions of Proposition 3 are satisfied, and the fundamental problem of dynamics has a unique – and stable – solution for any values of $X^*$, $Y^*_{1,2}$.

If inequality (3.8) holds but inequality (3.9) is reversed, the broken line is zigzag-shaped: the slope of its outer segments is positive, and that of the middle segment, negative. In that case, depending on the value of $X^*$, one has either a unique (and stable) solution or two stable solutions plus one unstable one (a cusp).

Note that the conclusion as to the number of equilibria of system (3.5) has nothing to do with the consideration of deformations; it may be derived from algebraic considerations [13]. As to the nature of the stability, it will be seen below that the situation depends on the choice of the deformation space.

## 4. THE CASE OF TWO CONSTRAINTS

If $k = 2$, the phase space of system (2.1) is four-dimensional. By analogy with Eqs (3.1), we write the equations of motion of the subsystem in the form

$$\begin{aligned} u_j' &= \Phi_j(u_j, v_j)v_j + \Psi_j(u_j, v_j)f_j(u_1, u_2), \quad v_j' = f_j(u_1, u_2) \\ f_j(u_1, u_2) &= G_j(t_0, q_1(u_1), q_2(u_2), q_i, 0, 0, \dot{q}_i); \quad j = 1, 2 \end{aligned} \tag{4.1}$$

The equilibrium positions of system (4.1) may be found by simultaneous solution of the equations

$$f_j(u_1, u_2) = 0, \quad f_2(u_1, u_2) = 0 \tag{4.2}$$

Let $(u_1^*, u_2^*)$ be some solution of system (4.2). The Jacobi matrix is

$$\mathbf{A} = \|a_{ij}\| = \|\partial f_i(u_1^*, u_2^*)/\partial u_j\| \tag{4.3}$$

*Proposition* 6. If the conditions

$$a_{11} > 0, \quad a_{22} > 0, \quad \det\mathbf{A} > 0, \quad a_{12}a_{21} \geq 0 \tag{4.4}$$

are satisfied, then $(\mathbf{u}^*, 0)$ is stable equilibrium position of system (4.1). But if at least one of the conditions (1) $\det\mathbf{A} < 0$, (2) $a_{11} < 0$, $a_{22} < 0$ is satisfied, that equilibrium is unstable.

*Proof.* Linearizing system (4.1) in the neighbourhood of the equilibrium, we obtain

$$\left\| \begin{matrix} \mathbf{u} \\ \mathbf{v} \end{matrix} \right\|' = \left\| \begin{matrix} \mathbf{\Psi A} & \mathbf{\Phi} \\ \mathbf{A} & \mathbf{0} \end{matrix} \right\|, \quad \mathbf{\Psi} = \operatorname{diag}\{\Psi_1, \Psi_2\}, \quad \mathbf{\Phi} = \operatorname{diag}\{\Phi_1, \Phi_2\} \tag{4.5}$$

where the values of the functions $\Psi_j$ and $\Phi_j$ are evaluated at the equilibrium position.

We introduce the notation

$$\chi_0 = \Phi_1\Phi_2, \ \chi_1 = \Psi_1\Phi_2 + \Psi_2\Phi_1, \ \chi_2 = \Psi_1\Psi_2, \ \chi_3 = a_{11}\Phi_1 + a_{22}\Phi_2, \ \chi_4 = a_{11}\Psi_1 + a_{22}\Psi_2$$

The characteristic equation for system (4.5) has the form

$$\begin{aligned} &p_4\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0 \\ &p_4 = 1, \ p_3 = -\chi_4, \ p_2 = \chi_2\det\mathbf{A} - \chi_3, \quad p_1 = \chi_1\det\mathbf{A}, \ p_0 = \chi_0\det\mathbf{A} \end{aligned} \tag{4.6}$$

It is well known [14] that the roots of Eq. (4.6) lie in the left half-plane if and only if

$$p_s > 0, \quad s = 0, 1, 2, 3, 4; \quad p_4p_1^2 - p_1p_2p_3 + p_0p_3^2 < 0 \tag{4.7}$$

We will first prove the case of instability. If $\det\mathbf{A} < 0$, then $p_0 < 0$, but if $a_{11} < 0$, $a_{22} < 0$, then $p_3 < 0$ irrespective of the dependence on the values of $\Psi_s < 0$, $\Phi_s < 0$. In both cases, the first group of inequalities in (4.7) fails to hold.

To prove stability, we observe that the first three conditions of (4.4) automatically imply the truth of the first group of inequalities (4.7), while the last inequality, in view of (4.6), may be written as follows:

$$\chi_1^2 \det\mathbf{A} < -\chi_0\chi_1\chi_3 \det\mathbf{A} + \chi_3(a_{11}\Psi_1^2\Phi_2 + a_{22}\Phi_1\Psi_2^2) \tag{4.8}$$

Since $a_{12}a_{21} > 0$, it follows that $\det\mathbf{A} < a_{11}a_{22}$. Replacing $\det\mathbf{A}$ on the left of (4.8) by $a_{11}a_{22}$ and dropping the first term on the right (which is always positive), we obtain the stronger condition

$$\chi_1^2 a_{11}a_{22} < \chi_3(a_{11}\Psi_1^2\Phi_2 + a_{22}\Phi_1\Psi_2^2) \tag{4.9}$$

Taking the expressions for $\chi_1$ and $\chi_3$ into consideration, removing parentheses in (4.9) and collecting like terms, we arrive at the inequality

$$2a_{11}a_{22}\Psi_1\Psi_2 < a_{11}^2\Psi_1^2 + a_{22}^2\Psi_2^2 \tag{4.10}$$

the truth of which follows from the theorem on the relationship between arithmetic and geometric means (generally speaking, inequality (4.10) may be replaced by an equality, but in that case the weaker inequality (4.8) holds in the strict sense). This proves the proposition.

*Remark* 1. As in the case considered in the previous section, that of a single friction pair, the assumptions of Proposition 6 relate only the dynamic characteristics of the system and are invariant to the viscoelastic properties of the contact pairs. We shall show that, if the matrix **A** satisfies the first three conditions of (4.4), but $a_{12}a_{21} < 0$, or $\Delta > 0$, $a_{11}a_{22} < 0$, then the equilibrium may be either stable or unstable, depending on the coefficients $\Psi_s$ and $\Phi_s$.

Consider the first of these cases. The first term on the right of inequality (4.8) is of degree 4 in the coefficients $\Phi_s$, but the other terms are only of degree 2. Consequently, the truth of the inequality may be achieved by taking $\Phi_1 = \Phi_2$ to be sufficiently large numbers. On the other hand, if these numbers are close to zero, the principal term on the right of inequality (4.8) will be the second. If $a_{11}\Psi_1 = a_{22}\Phi_2$, then inequality (4.9) becomes an equality, and so, by virtue of our assumption $a_{12}a_{21} < 0$, condition (4.8) fails to hold.

If the second case, assigning a sufficiently large value to whichever of the coefficients $\Phi_s$ corresponds to the negative element $a_{ss}$ we obtain $p_3 < 0$, indicating instability. Conversely, if $\Phi_s$ is sufficiently small, the stability conditions (4.7) are satisfied.

*Remark* 2. Suppose the first three conditions (4.4) are satisfied, but the fourth is not. Then the first group of inequalities (4.7) will hold, but the truth of the last inequality will depend on the numbers $\Phi_s$ and $\Psi_s$, which characterize the viscoelastic properties of the deformations. Varying these numbers, one can modify the nature of the stability. This modification is accompanied by the appearance of a pair of pure imaginary roots of the characteristic equation (4.6), indicating bifurcation – the birth of a cycle – in accordance with the Poincaré–Andronov–Hopf scenario.

For systems with three or more friction pairs, results analogous to Proposition 6 have not yet been established. The following propositions relating to the case under consideration are less general in nature.

*Proposition* 7. If $A$ is a symmetric positive-definite matrix, the equilibrium position of system (4.1) is asymptotically stable for any negative numbers $\Phi_s$ and $\Psi_s$.

The validity of this statement follows from the Kelvin–Tait–Chetayev theorem.

*Proposition* 8. If at least one of the following two conditions is satisfied

$$1)\ \det\mathbf{A} < 0,\ 2)\ a_{ss} < 0,\ s = 1, \dots, k$$

the equilibrium position of system (4.1) is unstable for any negative numbers $\Phi_s$ and $\Psi_s$.

The proof is analogous to that of Proposition 6.

## 5. ANALYSIS OF THE PAINLEVÉ–KLEIN SYSTEM

Let us consider the Painlevé–Klein example considered above under different physical assumptions: the length of the rod is fixed, and the guides are deformable. We shall assume that $m_1 = m_2 = 1/2$. The equations of motion of the rod may be written as follows [13]:

$$\ddot{x} = -\mu_1|N_1| - \mu_2|N_2| + X, \quad \ddot{y} = -N_1 + N_2 + Y$$
$$k^2\ddot{\varphi} = h(\mu_1|N_1| - \mu_2|N_2|) - b(N_1 + N_2), \quad b = l\cos\varphi \neq 0 \tag{5.1}$$

where $k = l$ is the radius of inertia of the rod, $2h$ is the distance between the guides, $x$ and $y$ are the coordinates of the centre of mass, $\varphi$ is the angle between the rod and the guides, and $X$, $Y$ and $M$ are the applied forces and the torque.

We introduce the coordinates $q_{1,2} = h - l\sin\varphi \mp y$, which represent the local deformations of the guides. Ignoring the variations of the variables $y$ and $\varphi$ in the equations of motion, we deduce from Eqs (5.1) that

$$\ddot{q}_j = (-1)^j(N_2 - N_1 + Y) + \frac{b}{k^2}(b(N_1 + N_2) - M) - \frac{bh}{k^2}(\mu_1|N_1| - \mu_2|N_2|), \quad j = 1, 2 \tag{5.2}$$

The equilibrium positions of system (5.2) may be found by equating the right-hand sides to zero. Adding and subtracting the equations of the system, we obtain

$$N_1 = N_2 + Y, \quad 2\kappa N_2 - \mu_1|N_2 + Y| + \mu_2|N_2| = M^*$$
$$M^* = M/h - \kappa Y, \quad \kappa = b/h = \operatorname{ctg}\varphi \tag{5.3}$$

Since the second equation of (5.3) contains two absolute value symbols, it follows that, depending on the coefficients of friction and the quantities $Y$ and $M$, that the number of solutions varies from zero to three [13]. The matrix (4.3) is

$$\mathbf{A} = \sin^2\varphi \begin{Vmatrix} 1 + 2\kappa^2 - \nu_1 & -1 - \nu_2 \\ -1 - \nu_1 & 1 + 2\kappa^2 - \nu_2 \end{Vmatrix}$$
$$\nu_j = (-1)^{(j+1)}\kappa s_j\mu_j, \quad s_j = \operatorname{sign}N_j, \quad j = 1, 2 \tag{5.4}$$

The determinant of $\mathbf{A}$ is equal to

$$\det\mathbf{A} = 2(2\kappa^2 - \nu_1 - \nu_2)\sin^2\varphi \tag{5.5}$$

Proposition 6, applied to the matrix (5.4), implies the following results: if the conditions

$$\nu_1 + \nu_2 < 2\kappa^2, \quad \nu_{1,2} < 1 + 2\kappa^2 \; (1 + \nu_1)(1 + \nu_2) > 0 \tag{5.6}$$

are satisfied, the equilibrium is asymptotically stable; but if

$$\nu_1 + \nu_2 > 2\kappa^2 \tag{5.7}$$

it is unstable.

In the plane of the parameters $\nu_1$, $\nu_2$, the domain (5.6) is the union of a triangle and a rectangular sector, while (5.7) is a half-plane (the unstable domains are shown in Fig. 1 by double hatching and the stable domains by single hatching).

In view of the relations

$$\mu_j = |\nu_j|/\kappa, \quad j = 1, 2$$

the plane is divided by the four straight lines $\pm\nu_1 \pm \nu_2 = 2\kappa^2$ (the dotted lines in Fig. 1) into parts that differ in the number of solutions of system (5.3).

1. The interior of the square

$$|\nu_1| + |\nu_2| < 2\kappa^2 \tag{5.8}$$

corresponds to the regular case: system (5.3) has a unique solution for any $Y$ and $M^*$ [13]. Depending on the slope of the rod $\varphi$, there are two possibilities:

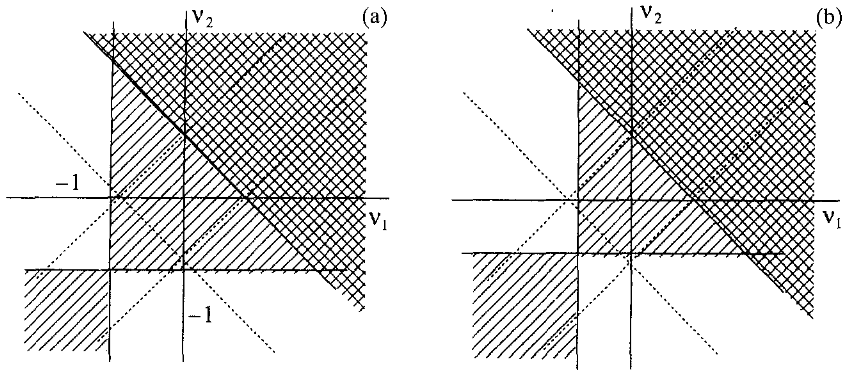


Fig. 1

(1) If

$$\varphi \geq \operatorname{arctg}\sqrt{2} \approx 54.7^\circ \tag{5.9}$$

the square (5.8) is in the interior of the stable domain (5.6) (Fig. 1a).

(2) If inequality (5.9) is reversed, part of the aforementioned square is in the stable domain (5.6) and part in a domain where the nature of the stability depends on the form of the functions (1.5) (Fig. 1b). Note that the change in the nature of the stability is associated in this case with bifurcation and the birth of a cycle, since the characteristic equation (4.6) has a zero root only at points of the straight line $\nu_1 + \nu_2 = 2\kappa^2$.

2. If

$$||\nu_1| - |\nu_2|| > 2\kappa^2 \tag{5.10}$$

system (5.3), depending on $Y$ and $M^*$, has two solutions or none at all [6]. Condition (5.10) is satisfied in the four rectangular sectors formed by the continuations of the sides of the square (5.8).

To fix our ideas, let us assume that $\mu_1 > \mu_2 + 2\kappa$ (that is, we are considering the left and right sectors). It can be verified that if system (5.3) has two solutions, they differ in the sign of $\nu_1$. One of the points corresponding to them in the $(\nu_1, \nu_2)$ plane lies in the right sector (instability) and the other, in the left sector (stability or possible stability).

3. If inequalities (5.8) and (5.10) are reversed, system (5.3) may have one or three solutions [13]. Corresponding to this case, in the $(\nu_1, \nu_2)$ plane are the four half-strips adjacent to the sides of the square (5.8). Analysis of the second equation of (5.3) shows that, if the solution is unique, the corresponding point lies in the second, third or fourth quadrant. In the case of three solutions, one of them is represented by a point in the first quadrant, the other two, by points in the second and fourth quadrants. Thus, there are two realistic solutions of the equations of motion. In order to determine which of them actually occurs, the initial deformations must be taken into consideration.

## 6. EXAMPLE OF AN OSCILLATORY SOLUTION

Let us consider the braking of a rotating disk by means of two identical brake-shoes mounted at an angle $\alpha$ to one another (Fig. 2). The coordinates $q_1$ and $q_2$ are defined as minus the normal deformations of the shoes at the points of contact. External forces are assumed to press the disk to the shoes, so that these quantities remain negative.

The theorem on the motion of the centre of mass in the projections onto directions perpendicular to the shoes implies the equations

$$m\ddot{q}_1 = N_1 + N_2(\mu\sin\alpha - \cos\alpha) + F_1, \quad m\ddot{q}_2 = N_2 - N_1(\mu\sin\alpha + \cos\alpha) + F_2 \tag{6.1}$$

where $m$ is the mass of the disk, $\mu$ is the coefficient of friction, $F_1$ and $F_2$ are the projections of the principal vector of external forces, and $N_1$ and $N_2$ are the normal reactions of the shoes.

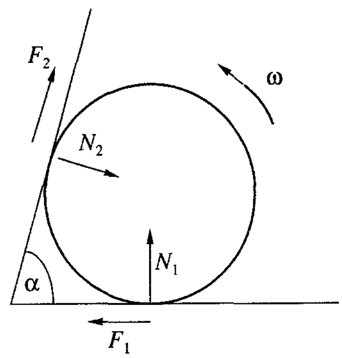


Fig. 2

The matrix (4.3) has the form

$$A = \frac{1}{m}\begin{Vmatrix} 1 & \mu\sin\alpha - \cos\alpha \\ -\mu\sin\alpha - \cos\alpha & 1 \end{Vmatrix} \tag{6.2}$$

Since the diagonal elements and determinant of the matrix (6.2) are positive, system (6.1) has a unique equilibrium position [13]. Conditions (4.4) reduce to the inequality

$$\mu < |\mathrm{ctg}\,\alpha| \tag{6.3}$$

Inequality (6.3) guarantees the stability of the equilibrium irrespective of the dependence on the specific form of the stresses as functions of the deformations. If $\alpha$ is a right angle, the inequality has no solutions. The inequality holds for obtuse and acute angles at sufficiently small values of the coefficient of friction.

The case of a right angle $\alpha$ merits a more detailed discussion. Since condition (6.3) fails to hold, the form of the functions (1.5) must be taken into consideration to solve the problem of stability. The first group of inequalities (4.7) holds for any admissible values of $\Phi_j$ and $\Psi_j$ ($j = 1, 2$), but the last inequality may or may not hold. If that inequality becomes an equality, this corresponds to a bifurcation and the birth of a cycle. The frequency of the periodic motion born in the bifurcation may be estimated from the first approximation, but determination of the direction of the bifurcation and the amplitude requires a non-linear analysis [15].

Let us assume that in equilibrium $\Phi_j = \Phi$, $\Psi_j = \Psi$ ($j = 1, 2$). Then the stability conditions (4.7) reduce to the inequality

$$-\mu^2\Phi < \Psi^2(1 + \mu^2) \tag{6.4}$$

Reversal of this inequality indicates bifurcation and birth of a cycle, the frequency of the periodic motion then being $\omega = -(\mu^2 + 1)\Psi/\mu$.

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